Thermal noise and stochastic strings in AdS/CFT

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Thermal noise and stochastic strings in AdS/CFT

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Abstract: We clarify the structure of thermal noise in AdS/CFT by studying the dynamics of an equilibrated heavy quark string. Using the Kruskal extension of the correspondence to generate the dynamics of the field theory on the Keldysh contour, we show that the motion of the string is described by the classical equations of motion with a stochastic boundary condition on the stretched horizon. The form of the stochastic boundary condition is consistent with the dissipation on this surface and is found by integrating out the fluctuations inside of the stretched horizon. Solving the equations of motion for the fluctuating string we determine the full frequency dependence of the random force on the boundary quark and show that it is consistent with the frequency dependent dissipation. We show further that the stochastic motion reproduces the bulk to bulk two point functions of the Kruskal formalism. These turn out to be related to the usual retarded bulk to bulk propagator by KMS relations. Finally we analyze the stochastic equations and give a bulk picture of the random boundary force as a flip-flopping trailing string solution. The basic formalism can be applied to the fluctuations of gravitons, dilatons, and other fields.

Keywords: AdS-CFT Correspondence, Thermal Field Theory, Black Holes

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1 Introduction

The AdS/CFT correspondence has been used extensively to study the properties of strongly coupled gauge theories in a controllable setting [1–3]. The correspondence has led to important insights into the nature of strongly coupled plasmas. In particular, refs. [4–8] established that in a large class of gauge theories with gravity duals the ratio between the shear viscosity and the entropy is

$$\frac{\eta}{s} = \frac{\hbar}{4\pi}\,. \tag{1.1}$$

This was important because it showed that there exist certain theories which realize the small transport time scales needed to explain the elliptic flow observed at RHIC [9–16]. Since this work on shear viscosity many other transport properties of strongly coupled plasmas have been computed using the correspondence. Of particular relevance to this

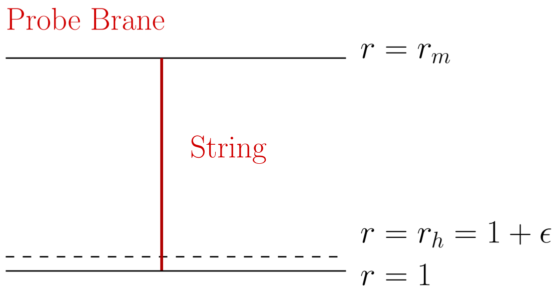

Figure 1. A schematic of a classical string in AdS_5 corresponding to a heavy quark. The horizon is at $r = 1$ in the coordinates of this work. The stretched horizon is at $r_h = 1 + \epsilon$ and the endpoint of the string is at r_m with $r_m \gg 1$. Gravity pulls downward in this figure.

work is the computation of the heavy quark drag and diffusion coefficient in $\mathcal{N} = 4$ Super Yang Mills (SYM) at large N_c and strong coupling [17–19].

Most of the time, thermal noise is neglected in AdS/CFT. This seems at odds with the fluctuation-dissipation theorem and leads to some seemingly incorrect results from the correspondence. For instance, it predicts the absence of long-time hydrodynamic tails [20], zero drag on mesons [21–23], and the lack of Brownian motion of a quark string in AdS_5 [17]. In many cases, the effect of thermal noise is suppressed either by large N or large λ, and therefore these inconsistencies with field theory intuition were rationalized as an artifact of these restrictive limits. Certain transport properties such as meson transport [24] and momentum diffusion [18, 25, 26] are intrinsically related to the fluctuations. These transport rates were computed using the correspondence by computing the drag and appealing to the boundary fluctuation-dissipation theorem to determine the fluctuation rate. A notable exception to this rule is the calculation of the momentum broadening of a fast heavy quark [25, 26] which used the Kruskal formalism to compute this diffusion rate in an out of equilibrium setting where the fluctuation dissipation theorem does not apply [25].

The purpose of the present work is to overcome these difficulties by working through the simplest possible system which should exhibit drag and noise in AdS/CFT. This system is the Brownian motion of a heavy quark placed in the $\mathcal{N} = 4$ SYM plasma. A quark in AdS/CFT is represented as an open string stretching from the horizon up to a probe brane. A schematic of the AdS geometry together with the heavy quark probe and the stretched horizon (see below) is shown in figure 1. The various problems with large N_c and large λ are easily summarized by the simple fact that, according to the classical equations of motion, the string does not move in the absence of external force. Clearly this is not the gravity dual of a Brownian particle in equilibrium with plasma.

One expects that the black brane should emit Hawking radiation inducing random motion on the string [27]. Hawking radiation is consistent with the fluctuation-dissipation theorem [28–33], and this consistency should lead to the correct picture of drag and diffusion of a heavy quark in the boundary theory. This idea has been sugested by several authors but was never clarified [34, 35]. When analyzing dissipation in the context of black holes, the notion of a stretched horizon plays a central role [6, 36, 37] and this surface is shown in figure 1 at $r = r_h = 1+\epsilon$. It will turn out that integrating out the fluctuations within the stretched horizon will yield a stochastic equation of motion with the required noise at the horizon.

To keep this work self contained we have provided a fairly extensive review of the necessary ingredients. Section 2 reviews the contour formalism of thermal field theory and shows how Brownian motion arises naturally from this algebraic structure. Section 4.1 reviews AdS/CFT in the Kruskal plane and generalizes the Kruskal/Keldysh correspondence to $\sigma = 0$ and the ra basis which is more natural for the classical AdS/CFT setup. Section 5.1 shows that bulk to bulk contour correlation functions respect KMS relations. This result which, while known [29, 30], was not entirely appreciated. Finally, section 5.2 presents a derivation of the stochastic force on the horizon and the subsequent section analyzes the results. A readable summary of the results and the bulk picture of Brownian motion is presented in section 6.

2 Langevin dynamics

2.1 Notation and overdamped motion

In this section we will review briefly the Langevin process. Consider a heavy particle moving with velocity $\dot{x}$, subjected to drag $-\eta\dot{x}$ and random noise ξ

$$M_{\rm kin}\frac{d^2x}{dt^2} + \eta\frac{dx}{dt} = \xi\,, \qquad \big\langle \xi(t)\xi(t')\big\rangle = 2T\eta\,\delta(t-t')\,. \tag{2.1}$$

Here and below we will only write the equation of motion for the x component of the motion, $\xi \equiv \xi_x$. We also have anticipated that the strength of the noise $\langle\xi(t)\xi(t')\rangle$ is related to the drag η through the fluctuation dissipation theorem. $M_{\rm kin}$ is the quasi-particle mass including in-medium modification of the mass. The diffusion coefficient can be written as

$$D = \frac{T}{\eta}\,. \tag{2.2}$$

Let us recall how this formula follows. Consider the ultimate long time limit. In this limit the $-M_{\rm kin}\omega^2$ term may be dropped since it is proportional to frequency squared and the motion is overdamped

$$\eta\frac{dx}{dt} = \xi\,, \tag{2.3}$$

i.e. the dissipation exactly balances the force. Solving for the position of the quark, squaring, and averaging, we find the squared displacement

$$\big\langle x^2(t)\big\rangle = 2\frac{T}{\eta}t\,. \tag{2.4}$$

In the diffusion equation a Gaussian drop of dye spreads out as $\big\langle x^2(t)\big\rangle = 2Dt$ leading to the identification in eq. (2.2). This overdamped limit where the drag force exactly balances the velocity will be central in discussing the membrane paradigm and black holes.

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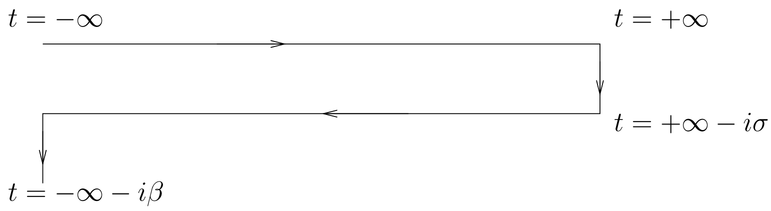

Figure 2. The Schwinger Keldysh contour. The fields labeled by "1" live on the upper time ordered axis, while the fields labeled by "2" live on the lower anti-time ordered axis.

2.2 Langevin dynamics from the contour

Next we review how the Langevin equations can be derived from the real time path integral [38, 39]. The purpose here is not to track down every intricacy as there are reviews for this purpose [40]. In the recent literature we have found refs. [41, 42] instructive. For a heavy particle coupled to an equilibrated bath of forces the real time partition function is

$$Z = \left\langle \int [\mathrm{D}x_1][\mathrm{D}x_2]\, e^{i\int \mathrm{d}t_1 M_Q^o \dot{x}_1^2}\, e^{-i\int \mathrm{d}t_2 M_Q^o \dot{x}_2^2}\, e^{i\int dt_1 \mathcal{F}_1(t_1)x_1(t_1) - i\int dt_2 \mathcal{F}_2(t_2)x_2(t_2)} \right\rangle_{\text{bath}}, \tag{2.5}$$

where the path integration is along the Schwinger-Keldysh contour [41, 43–45]. The path integral over the "1" type coordinates (the upper line in figure 2) represents the amplitude, while the path integral over the "2" type coordinates (the lower line in figure 2) represents the conjugate amplitude. The path integral over the vertical pieces of the contour (which is not explicitly written in eq. (2.5) but is implied by figure 2) represents the average over the thermal density matrix $e^{-\beta H}$. The choice of σ is arbitrary and reflects the fact that the thermal density matrix is stationary and therefore the average can be performed either at future or past infinity. An analogous dichotomy exists in the gravitational setup as discussed in section 4.1. We have not written the path integral for the bath which is also evaluated along the contour.

When the particle is very heavy, the action is large and the motion is quasi-classical. The medium forces are small compared to the inertial terms and we can expand to second order, average over the bath, and re-exponentiate to find

$$Z = \int [\mathrm{D}x_1][\mathrm{D}x_2] e^{i\int \mathrm{d}t_1 M_Q^o \dot{x}_1^2}\, e^{-i\int \mathrm{d}t_2 M_Q^o \dot{x}_2^2} e^{-\frac{1}{2}\int \mathrm{d}t\mathrm{d}t' x_s(t)[\langle \mathcal{F}(t)\mathcal{F}(t')\rangle]_{ss'} x_{s'}(t')}\,. \tag{2.6}$$

Here the matrix of contour ordered correlation functions is

$$\left[\langle \mathcal{F}(t)\mathcal{F}(t')\rangle\right]_{ss'} \equiv i \begin{bmatrix} G_{11}(t,t') & -G_{12}(t,t') \\ -G_{21}(t,t') & G_{22}(t,t') \end{bmatrix}, \tag{2.7}$$

where for instance $G_{12}(t,t') = \langle F_1(t)F_2(t')\rangle$ is the average of the forces over the partition function of the bath. The relation to the operator formalism is the following: we define the time dependent operators

$$\hat{\mathcal{F}}_1(t) = e^{iHt}\hat{\mathcal{F}}(0)e^{-iHt}\,, \qquad \hat{\mathcal{F}}_2(t) = e^{+iH(t-i\sigma)}\hat{\mathcal{F}}(0)e^{-iH(t-i\sigma)}\,, \tag{2.8}$$

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and then the correlation functions are

$$iG_{11}(t,t') = \left\langle T\,\hat{\mathcal{F}}_1(t)\,\hat{\mathcal{F}}_1(t')\right\rangle , \tag{2.9a}$$
$$iG_{12}(t,t') = \left\langle \hat{\mathcal{F}}_2(t')\,\hat{\mathcal{F}}_1(t)\right\rangle , \tag{2.9b}$$
$$iG_{21}(t,t') = \left\langle \hat{\mathcal{F}}_2(t)\,\hat{\mathcal{F}}_1(t')\right\rangle , \tag{2.9c}$$
$$iG_{22}(t,t') = \left\langle \tilde{T}\,\hat{\mathcal{F}}_2(t)\,\hat{\mathcal{F}}_2(t')\right\rangle . \tag{2.9d}$$

The KMS condition relates the different time orderings so there is really only one independent function which can be taken to be the retarded Green function

$$iG_R(t) = \theta(t)\left\langle [\hat{\mathcal{F}}(t), \hat{\mathcal{F}}(0)]\right\rangle_{\rm bath} . \tag{2.10}$$

Using completeness and KMS relations it can be shown that

$$iG_{11}(\omega) = +\,i\mathrm{Re}\,G_R(\omega) - (1+2n)\mathrm{Im}\,G_R(\omega)\,, \tag{2.11a}$$
$$iG_{22}(\omega) = -\,i\mathrm{Re}\,G_R(\omega) - (1+2n)\mathrm{Im}\,G_R(\omega)\,, \tag{2.11b}$$
$$iG_{12}(\omega) = -\,2ne^{\omega\sigma}\mathrm{Im}\,G_R(\omega)\,, \tag{2.11c}$$
$$iG_{21}(\omega) = -\,2(1+n)e^{-\omega\sigma}\mathrm{Im}\,G_R(\omega)\,. \tag{2.11d}$$

Hidden in these relations is the inter-relation between the drag and noise. Almost all applications of real time thermal field theory have relied upon a rewritten version of the path integral known as the ra formalism (see, *e.g.*, refs. [41, 45–47]). (Of particular note is the next to leading order computation of the heavy quark diffusion coefficient in weakly coupled $\mathcal{N}=4$ SYM which relied exclusively on the ra setup [46].) Here and below we take $\sigma = 0$ and the average over the initial density matrix is entirely in the past. Since in a quasi classical limit the amplitude is not very different from the conjugate amplitude we define the retarded (r) and advanced fields (a) for the particle and also for the forces

$$x_r = \frac{x_1+x_2}{2}\,, \qquad x_a = x_1 - x_2\,, \qquad \mathcal{F}_r = \frac{\mathcal{F}_1+\mathcal{F}_2}{2}\,, \qquad \mathcal{F}_a = \mathcal{F}_1 - \mathcal{F}_2\,. \tag{2.12}$$

x_a should be considered a small parameter in the classical limit [47]. Effecting this transformation we find

$$Z = \int [\mathrm{D}x_r][\mathrm{D}x_a] e^{-i\int \mathrm{d}t\, M_Q^o x_a \ddot{x}_r - \int \mathrm{d}t\mathrm{d}t' x_a(t) iG_R(t,t') x_r(t') - \frac{1}{2}x_a(t)G_{\rm sym}(t,t')x_a(t')}\,. \tag{2.13}$$

We have defined the propagators

$$G_{\rm sym}(t,t') = \left\langle \mathcal{F}_r(t)\mathcal{F}_r(t')\right\rangle = \frac{1}{2}\left\langle \{\hat{\mathcal{F}}(t), \hat{\mathcal{F}}(t')\}\right\rangle , \tag{2.14a}$$
$$iG_R(t,t') = \left\langle \mathcal{F}_r(t)\mathcal{F}_a(t')\right\rangle = \theta(t)\left\langle [\hat{\mathcal{F}}(t), \hat{\mathcal{F}}(0)]\right\rangle , \tag{2.14b}$$
$$iG_A(t,t') = \left\langle \mathcal{F}_a(t)\mathcal{F}_r(t')\right\rangle = -\theta(-t)\left\langle [\hat{\mathcal{F}}(t), \hat{\mathcal{F}}(0)]\right\rangle , \tag{2.14c}$$

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made use of the familiar contour relationships

$$G_{\rm sym} = \frac{i}{4}\left[G_{11}+G_{22}+G_{12}+G_{21}\right], \tag{2.15a}$$
$$iG_R = \frac{i}{2}\left[G_{11}-G_{22}+G_{21}-G_{12}\right], \tag{2.15b}$$
$$iG_A = \frac{i}{2}\left[G_{11}-G_{22}-G_{21}+G_{12}\right], \tag{2.15c}$$
$$0 = G_{11}+G_{22}-G_{12}-G_{21}, \tag{2.15d}$$

and also have used the reality relation between the advanced and retarded propagators

$$G_A(t) = G_R(-t)\,, \qquad G_A(\omega) = G_R^*(\omega)\,. \tag{2.16}$$

The KMS relations become simply the canonical form of the fluctuation dissipation theorem

$$G_{\rm sym}(\omega) = -(1+2n){\rm Im}G_R(\omega)\,. \tag{2.17}$$

In Fourier space the path integral is simply

$$Z = \int [{\rm D}x_r][{\rm D}x_a] \exp\left(-i\int\frac{{\rm d}\omega}{2\pi}x_a(-\omega)[-M_Q^o\omega^2+G_R(\omega)]x_r(\omega)\right) e^{-\frac{1}{2}\int\frac{{\rm d}\omega}{2\pi}x_a(-\omega)[G_{\rm sym}(\omega)]x_a(\omega)}\,. \tag{2.18}$$

After Fourier transforming the Gaussian by introducing a noise variable,

$$e^{-\frac{1}{2}\int\frac{{\rm d}\omega}{2\pi}x_a(-\omega)[G_{\rm sym}(\omega)]x_a(\omega)} = \int[{\rm D}\xi]e^{+i\int x_a(-\omega)\xi(\omega)}\, e^{-\frac{1}{2}\frac{\xi(-\omega)\xi(\omega)}{G_{\rm sym}(\omega)}}\,, \tag{2.19}$$

the partition function reads

$$Z = \int[{\rm D}x_r][{\rm D}x_a][{\rm D}\xi]e^{-\frac{1}{2}\int\frac{{\rm d}\omega}{2\pi}\frac{\xi(-\omega)\xi(\omega)}{G_{\rm sym}(\omega)}} \times\exp\left(-i\int\frac{{\rm d}\omega}{2\pi}x_a(-\omega)\left[-M_Q^o\omega^2x_r(\omega)+G_R(\omega)x_r(\omega)-\xi(\omega)\right]\right)\,. \tag{2.20}$$

At this point one may integrate over $x_a(-\omega)$ yielding the path integral

$$Z = \int[{\rm D}x_r][{\rm D}\xi]e^{-\frac{1}{2}\int\frac{{\rm d}\omega}{2\pi}\frac{\xi(-\omega)\xi(\omega)}{G_{\rm sym}(\omega)}}\,\delta_\omega\left[-M_Q^o\omega^2x_r(\omega)+G_R(\omega)x_r(\omega)-\xi(\omega)\right]\,. \tag{2.21}$$

This equation means that the partition function is simply an average over the classical trajectories under the influence of a random colored force,

$$\left[-M_Q^o\omega^2+G_R(\omega)\right]x(\omega) = \xi(\omega)\,, \qquad \langle\xi(-\omega)\xi(\omega)\rangle = G_{\rm sym}(\omega) = -(1+2n)\,{\rm Im}G_R(\omega)\,. \tag{2.22}$$

In time this equation reads

$$M_Q^o\frac{d^2x}{dt^2} + \int^t {\rm d}t'\,G_R(t,t')\,x(t') = \xi(t)\,, \qquad \left\langle\xi(t)\xi(t')\right\rangle = G_{\rm sym}(t,t')\,, \tag{2.23}$$

which is a generalized Langevin equation with the drag and corresponding noise [48, 49]. One method to implement such colored noise on the computer has been given in ref. [50].

At small frequencies we can expand the retarded Green function to ω^2

$$G_R(\omega) = -i\omega\eta - \Delta M\omega^2\,, \tag{2.24}$$

and then the effective equation of motion is the original Langevin equation

$$M_{\rm kin}\frac{d^2x(t)}{dt^2} + \eta\frac{dx(t)}{dt} = \xi(t)\,, \qquad \left\langle \xi(t)\xi(t')\right\rangle = 2T\eta\,\delta(t-t')\,, \tag{2.25}$$

where we have defined the kinetic mass as $M_{\rm kin}(T) = M_Q^o + \Delta M$.

3 Review of trailing strings

Our purpose here is to collect some of the results from the AdS/CFT correspondence on the drag and diffusion of heavy quarks [17–19]. A canonical choice of coordinates for the metric of black hole AdS$_5$ (which we will denote with bars) is

$$ds_5^2 = \frac{\bar{r}^2}{L^2}\left[-f(b\bar{r})dt^2 + d\mathbf{x}^2\right] + \frac{L^2 d\bar{r}^2}{f(b\bar{r})\bar{r}^2}\,, \tag{3.1}$$

with $f(r) = 1 - 1/r^4$. Here b is the inverse horizon radius which is related to the Hawking temperature, $b = 1/\pi T L^2$. We will use a different set of conventions defining $r \equiv b\bar{r}$, such that r is a measure of energy in units of πT

$$ds^2 = (\pi T)^2 L^2\left[-r^2 f(r)dt^2 + r^2 d\mathbf{x}^2\right] + \frac{L^2 dr^2}{f(r)r^2}\,. \tag{3.2}$$

We are considering the dynamics of very massive quark which is represented in AdS$_5$ by a long string stretching from the horizon upwards towards the AdS boundary terminating at r_m as illustrated in figure 1. This straight string is a solution to the classical Nambu-Goto equations of motion. Then we consider small fluctuations around this long straight string.

The action for these fluctuations is (see appendix A)

$$S = -\int \mathrm{dtdr}\left[m + \frac{1}{2}T_o(r)(\partial_r x)^2 - \frac{m}{2f}(\partial_t x)^2\right]\,, \tag{3.3}$$

where

$$T_o(r) = \frac{(\pi T)^3 L^2}{2\pi\ell_s^2}\,fr^4 = \left(\sqrt{\lambda}\pi^2 T^3/2\right)\,fr^4\,, \tag{3.4}$$

has the meaning of the local tension, and

$$m = \frac{(\pi T)L^2}{2\pi\ell_s^2} = \frac{\sqrt{\lambda}T}{2}\,, \tag{3.5}$$

is mass per unit r. The zero temperature mass of the quark $M_Q^o = mr_m$. The speed of waves on the string is $c_s = \sqrt{T_o(r)f/m} = \pi T f r^2$ and therefore waves propagate from $r_h = 1+\epsilon$ to the boundary in a time of order $\sim 1/\pi T\log(1/\epsilon)$.

For a prescribed boundary value $x_o(\omega)$ we can solve for the classical waves on the string by imposing retarded boundary conditions at the horizon. The classical equations of motion after Fourier transforming in time are

$$\partial_r \big(T_o(r) \partial_r x(\omega, r) \big) + \frac{m\omega^2}{f} x(\omega, r) = 0\,. \tag{3.6}$$

We will define $F_\omega(r)$ as the retarded boundary to bulk propagator, *i.e.* the solution which satisfies

$$\lim_{r \to r_m} F_\omega(r) = 1\,,$$

and retarded boundary conditions.

The waves on the string due to the sinusoidal motion of the end point of the string are

$$x_o(\omega, r) = x_o(\omega) F_\omega(r) = x_o(\omega) + \frac{-i\omega x_o(\omega)}{(2\pi T)} \left[\tan^{-1}(z) - \tanh^{-1}(z) \right] + O(\omega^2)\,, \tag{3.7}$$

with $z \equiv 1/r$. The term multiplying the velocity $-i\omega x_o(\omega)$ is the "trailing string" solution of refs. [17, 19]. (In writing this equation we have assumed that we are not exponentially close to the horizon, *i.e.* $\omega \log(1/\epsilon) \ll \pi T$.) Thus to lowest order the string trails behind the sinusoidal motion according to the expected form. The next term in this series is proportional to the acceleration and is analyzed in appendix A.

The retarded force-force correlator is found by taking the boundary limit [18, 51]

$$G_R^o \equiv \lim_{r \to r_m} T_o(r) F_{-\omega} \partial_r F_\omega = -M_Q^o \omega^2 + G_R(\omega)\,, \tag{3.8}$$

where M_Q^o is the zero temperature quark mass, and the term $-M_Q^o \omega^2$ arises from the "divergent" part of the boundary limit. Then using $T_o(r)$ given above and the retarded solution given in eq. (3.7) and appendix A, we find

$$G_R(\omega) = -i\omega\eta - \Delta M \omega^2\,, \tag{3.9}$$

with

$$\eta = \frac{1}{2}\sqrt{\lambda}\pi T^2\,, \tag{3.10}$$

$$\Delta M = -\frac{\sqrt{\lambda} T}{2}\,. \tag{3.11}$$

As discussed above, the field theory interpretation from this form for the retarded force-force correlator is that the quark obeys the stochastic motion in eq. (2.23) with the specified transport coefficient η (first computed in ref. [17–19]) and in medium mass shift ΔM (first computed in ref. [17]). The primary aim of this work is to show how this stochastic equation is derived in AdS/CFT and to give a bulk picture to the stochastic process.

Before taking up this enterprise we make the following technical notes about the solutions to the bulk equations of motion eq. (3.6):

1. The conjugate solution satisfies $F_\omega^* = F_{-\omega}$ and obeys advanced boundary conditions. When constructing solutions we note that $\mathrm{Im} F_\omega(r)$ obeys the same differential equation but is normalizable, *i.e.* $\mathrm{Im} F(r) \to 0$ as $r \to \infty$.

2. The imaginary part of the retarded Green function,

$$\mathrm{Im} G_R(\omega) = \frac{T_o(r)}{2i}\left[F_{-\omega}(r)\partial_r F_\omega(r) - F_\omega(r)\partial_r F_{-\omega}(r)\right] , \tag{3.12}$$

can be evaluated at any radius. This is because the term in square brackets is the Wronskian of the differential equation which depends on r in a particular way which precisely cancels the leading $T_o(r)$ factor. This fact will be used repeatedly in section 5.3.

3. When evaluating the evolution of perturbations in the bulk, it is often very useful to define the retarded bulk to bulk propagator. Many authors (e.g. refs. [52, 53]) have defined a Green function which is infalling at the horizon and normalizable at the boundary

$$G_{\mathrm{ret}}(\omega, r, \bar{r}) = \frac{\mathrm{Im}F_\omega(r)F(\bar{r})\theta(r,\bar{r}) + F_\omega(r)\mathrm{Im}F_\omega(\bar{r})\theta(\bar{r},r)}{T_o(\bar{r})W_{\mathrm{ret}}(\bar{r})} , \tag{3.13}$$

where we have defined the retarded Wronskian$\times T_o(r)$

$$\begin{aligned} T_o(r)W_{\mathrm{ret}}(r) &= T_o(r)\left[\mathrm{Im}F'(r)F(\bar{r}) - F'(r)\mathrm{Im}F(r)\right] , \\ &= -\mathrm{Im}G_R(\omega) . \end{aligned} \tag{3.14}$$

Note again this combination is proportional to the retarded force-force correlator and is independent of r.

4 The boundary picture of stochastic motion

4.1 AdS/CFT on the contour

In this section we review how the Schwinger-Keldysh formalism is constructed in AdS/CFT by using the full Kruskal structure of the black hole. Here we will extend the results of ref. [54] only slightly to show how the arbitrary σ of the contour is present in the Kruskal formalism. We will then specialize to $\sigma = 0$ and indicate how the results appear in the ra basis. The Kruskal/Keldysh correspondence was based on the pioneering work on Hawking radiation [28–33].

The Kruskal plane of the eternal black hole is exhibited in figure 3 (see appendix B for a summary of the Kruskal conventions adopted here.) Outside of the horizon there are two causally disconnected space time geometries both of which are asymptotically AdS. The right quadrant corresponds to the amplitude of the CFT while the left quadrant corresponds to the conjugate amplitude of the CFT. The dynamics of the two AdS/CFT's are coupled through boundary conditions at past and future infinity.

AdS/CFT in practice amounts to a four step procedure.

1. First determine a solution to the classical equations of motion with certain boundary values.

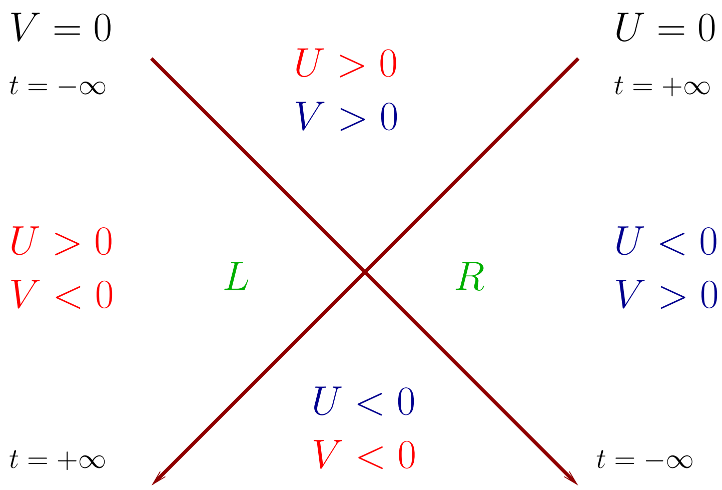

Figure 3. The full Kruskal plane. The right quadrant corresponds to the amplitude of the field theory (the "1" axis) while the left quadrant corresponds to the conjugate amplitude of the field theory (the "2" axis).

2. In order to uniquely specify this solution one must specify horizon boundary conditions.

3. This solution is then substituted into the classical action which reduces to a boundary term.

4. Finally, variation with respect to the boundary values determines the correlation functions of the CFT.

In the full Kruskal plane these standard AdS/CFT steps are the following.

1. First the equations of motion for the fluctuations on the string are solved in the full Kruskal plane with the boundary values

$$\lim_{r_1 \to r_m} x(\omega, r_1) = x_1^o(\omega)\,, \tag{4.1a}$$
$$\lim_{r_2 \to r_m} x(\omega, r_2) = x_2^o(\omega)\,. \tag{4.1b}$$

Here r_1 and r_2 are the radial coordinates in the right and left quadrants respectively. The general solutions in the right and left quadrants are

$$x(\omega, r_1) = a(\omega)F_\omega(r_1) + b(\omega)F_\omega^*(r_1)\,, \tag{4.2}$$
$$x(\omega, r_2) = c(\omega)F_\omega(r_2) + d(\omega)F_\omega^*(r_2)\,. \tag{4.3}$$

Since we have only two boundary constraints given by eq. (4.1), to uniquely specify the solution we must specify horizon boundary conditions.

2. The horizon boundary conditions specify the connection between the solution in the right quadrant and the solution in the left quadrant. We note that near the horizon the retarded and advanced solutions behave as

$$e^{-i\omega t}F_\omega(r_1) \sim e^{-i\frac{\omega}{2\pi T}\log(V)}\,, \tag{4.4}$$
$$e^{-i\omega t}F^*_\omega(r_1) \sim e^{+i\frac{\omega}{2\pi T}\log(-U)}\,. \tag{4.5}$$

The horizon conditions proposed in ref. [54] are that the solution should be analytic in the lower V plane and upper U plane. This physical choice is based on the intuition that if the right universe is to represent the time-ordered amplitude of the CFT, then the infalling mode (F_ω) should be positive energy (analytic in in lower V) while the outgoing mode (F^*_ω) should be negative energy (analytic in upper U). For instance in a free scalar theory the Green function at zero temperature is

$$iG_{11}(t,t') = \frac{1}{2E_{\mathbf{p}}}e^{-iE_{\mathbf{p}}(t-t')}\theta(t,t') + \frac{1}{2E_{\mathbf{p}}}e^{+iE_{\mathbf{p}}(t-t')}\theta(t',t)\,, \tag{4.6}$$

and therefore a one type source $J_1(t')$ will produce fields with positive energy wave at later time and induce negative energy wave at an earlier time. With this choice Herzog and Son analytically continue from the right quadrant ($V > 0$, $U < 0$) to the left quadrant ($V < 0$, $U > 0$) yielding from the solution in the right quadrant $x(\omega, r_1)$ the solution in the left quadrant

$$x(\omega, r_2) = a(\omega)\,e^{-\omega/2T}\,F_\omega(r_2) + b(\omega)\,e^{+\omega/2T}F^*_\omega(r_2)\,. \tag{4.7}$$

Now the values of $a(\omega)$ and $b(\omega)$ can be determined from the boundary values $x^o_1(\omega)$ and $x^o_2(\omega)$. The resulting solution ultimately reproduces the contour correlation functions for the specific choice of $\sigma = \beta/2$ [54].

Here we will generalize their work slightly by extending $V \to |V|e^{-i\theta}$ and $-U \to |U|e^{-i(2\pi-\theta)}$; in the original work $\theta = \pi$. With this choice for the analytic continuation of U and V, the radius is fixed since $(-U)V$ is changed ultimately by a factor $e^{-2\pi i}$ (note eq. (B.4)). We will find that this choice for the analytic continuation reproduces the contour relations for arbitrary σ. Loosely speaking, the difference between these choices is a decision about whether to perform the thermal average at past infinity $V = 0$ or future infinity $U = 0$. We do not have a sharper physical explanation for this choice at this point. With this analytic continuation the infalling and outgoing solutions behave as

$$F_\omega(r_1) \to e^{-\omega\sigma}F_\omega(r_2)\,, \tag{4.8}$$
$$F^*_\omega(r_1) \to e^{+\omega/T}e^{-\omega\sigma}F^*_\omega(r_2)\,, \tag{4.9}$$

where we have defined[1]

$$\sigma = \frac{\theta}{2\pi T}\,. \tag{4.10}$$

[1] Of course this definition will later turn out to correspond to the usual field theory definition.

The solution in eq. (4.2) when analytically continued to the left quadrant behaves as

$$x(\omega, r_2) = a(\omega)\, e^{-\omega\sigma} F_\omega(r_2) + b(\omega)\, e^{+\omega/T} e^{-\omega\sigma} F^*_\omega(r_2)\,. \tag{4.11}$$

Now as before we can solve for the coefficients $a(\omega)$ and $b(\omega)$ in terms of the $x^o_1(\omega)$ and $x^o_2(\omega)$ yielding the result

$$a(\omega) \,=\, x^o_1(\omega)(1 + n(\omega)) - x^o_2(\omega)e^{\omega\sigma} n(\omega)\,, \tag{4.12a}$$

$$b(\omega) \,=\, x^o_2(\omega)e^{\omega\sigma} n(\omega) - x^o_1(\omega)n(\omega)\,. \tag{4.12b}$$

The solution is now fully specified by its right and left quadrant solutions, eq. (4.2) and eq. (4.11).

3. Now we can substitute the solution into the boundary action

$$S_{\text{bndry}} = -\frac{T_o(r_m)}{2}\int_{r_1} \frac{\mathrm{d}\omega}{2\pi} x_1(-\omega, r_1)\partial_r x_1(\omega, r_1) + \frac{T_o(r_m)}{2}\int_{r_2} \frac{\mathrm{d}\omega}{2\pi} x_2(-\omega, r_2)\partial_r x_2(\omega, r_2)\,, \tag{4.13}$$

and determine the generating function

$$\begin{aligned} iS_{\text{bndry}} = -\frac{1}{2}\int \frac{\mathrm{d}\omega}{2\pi} & x^o_1(-\omega)\left[+i\mathrm{Re}G^o_R - (1 + 2n)\mathrm{Im}G^o_R\right] x^o_1(\omega) \\ & + x^o_2(-\omega)\left[-i\mathrm{Re}G^o_R - (1 + 2n)\mathrm{Im}G^o_R\right] x^o_2(\omega) \\ & - x^o_1(-\omega)\left[-2ne^{+\omega\sigma}\mathrm{Im}G^o_R\right] x^o_2(\omega) - x^o_2(-\omega)\left[-2(1+n)e^{-\omega\sigma}\mathrm{Im}G^o_R\right] x^o_1(\omega) \end{aligned} \tag{4.14}$$

Here we identified the retarded Green function with

$$G^o_R(\omega) \,=\, T_o(r)\frac{F_{-\omega}(r)\partial_r F_\omega}{|F(\omega, r)|^2}\bigg|_{r=r_m}\,. \tag{4.15}$$

4. Now by taking variations with respect to x^o_1 and x^o_2 one can reproduce the full contour correlation functions which obey the appropriate KMS relations in eq. (2.11).

4.2 Simple results in the ra setup

Before closing this review of ref. [54] let us show how these results work out in the ra basis. Specializing to $\sigma = 0$ and introducing the bulk r and a fields

$$x_r(\omega, r) = \frac{x_1(\omega, r) + x_2(\omega, r)}{2}\,, \qquad x_a(\omega, r) = x_1(\omega, r) - x_2(\omega, r)\,. \tag{4.16}$$

we find that the bulk fields are rather simply related to the boundary fields in the ra setup

$$x_a(\omega, r) \,=\, F^*_\omega(r)\, x^o_a(\omega)\,, \tag{4.17a}$$

$$x_r(\omega, r) \,=\, F_\omega(r)\, x^o_r(\omega) + i(1 + 2n)\,\mathrm{Im}F_\omega(r) x^o_a(\omega)\,. \tag{4.17b}$$

The boundary action in the ra formalism after rewriting eq. (4.13) is

$$S_{\text{bnry}} \,=\, -\frac{T_o(r_m)}{2}\int_{r_m} \frac{\mathrm{d}\omega}{2\pi}\,\left[x_a(-\omega, r)\partial_r x_r(\omega, r) + x_r(-\omega, r)\partial_r x_a(\omega, r)\right]\,. \tag{4.18}$$

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Substituting the solutions given in eq. (4.17) gives the desired result in the ra setup

$$iS_{\rm bnry} = -i\int \frac{\mathrm{d}\omega}{2\pi} x^o_a(-\omega)\left[G_R(\omega)\right] x^o_r(\omega) - \frac{1}{2}\int \frac{\mathrm{d}\omega}{2\pi} x^o_a(-\omega)\left[G_{\rm sym}(\omega)\right] x^o_a(\omega)\,, \tag{4.19}$$

with $G_{\rm sym}(\omega) = -(1+2n){\rm Im}G_R(\omega)$. Of course this could be found simply by changing variables in eq. (4.14), but it is hoped that the relatively simple causal relations in eq. (4.17) will be useful in understanding non-thermal thermal fluctuations of AdS/CFT in an out of equilibrium setting [55].

4.3 Stochastic motion on the boundary

Now we will show how these results lead directly to stochastic motion on the boundary. A formal path integral for the string is

$$Z = \int [\mathrm{D}x^o_1\, \mathbb{D}x_1]\, [\mathrm{D}x^o_2\, \mathbb{D}x_2]\, e^{iS_1 - iS_2}\,, \tag{4.20}$$

where in the right quadrant of the Kruskal plane, $\mathrm{D}x^o_1$ is a temporal path for the string endpoint, $\mathbb{D}x_1$ is a bulk path integral for the body of the string, and the analogous "2" symbols are defined for the left quadrant. Slightly more explicitly

$$[\mathrm{D}x^o_1] = \prod_t \mathrm{d}x^o_1(t)\,, \qquad [\mathbb{D}x_1] = \prod_{t,r} \mathrm{d}x(t,r)\,. \tag{4.21}$$

Imagine integrating out the bulk coordinates to find an effective action for the string endpoints. In a Gaussian approximation the integrals can be done. The result is a determinant times the exponential of the action evaluated with the classical solution which passes through the two endpoints and which obeys the appropriate horizon boundary conditions. This is of course simply the boundary action

$$S^o_{\rm eff} = -\frac{T_o(r_m)}{2}\int_{r_1} \frac{\mathrm{d}\omega}{2\pi} x_1(\omega,r_1)\partial_r x_1(\omega,r_1) + \frac{T_o(r_m)}{2}\int_{r_2} \frac{\mathrm{d}\omega}{2\pi} x_2(-\omega,r_2)\partial_r x_2(\omega,r_2)\,, \tag{4.22}$$

which can be evaluated using the results of the previous section

$$iS^o_{\rm eff} = -i\int \frac{\mathrm{d}\omega}{2\pi} x^o_a(-\omega)\left[-M^o_Q\omega^2 + G_R(\omega)\right] x^o_r(\omega) - \frac{1}{2}\int \frac{\mathrm{d}\omega}{2\pi}\, x^o_a(-\omega)\left[G_{\rm sym}(\omega)\right] x^o_a(\omega)\,. \tag{4.23}$$

Now the partition function for the string endpoint is

$$Z = \int [\mathrm{D}x^o_r \mathrm{D}x^o_a]\, e^{iS^o_{\rm eff}}\,. \tag{4.24}$$

At this point we recognize the same partition function as discussed in the introduction, and following the discussion after eq. (2.20) we conclude that the string endpoint obeys the expected equations of motion

$$\left[-M^o_Q\omega^2 + G_R(\omega)\right] x^o(\omega) = \xi^o(\omega)\,, \qquad \langle \xi^o(-\omega)\xi^o(\omega)\rangle = G_{\rm sym}(\omega) = -(1+2n)\,{\rm Im}G_R(\omega)\,. \tag{4.25}$$

5 The bulk picture of stochastic motion

In the previous section we integrated out all degrees of freedom of the string, except for the endpoints, to obtain the effective action for the endpoint coordinates. In this section we will integrate out only the degrees of freedom inside a stretched horizon. In this way we will obtain an equation of motion for a string with a friction and a noise acting on the string at the stretched horizon.

5.1 Bulk to bulk correlators in contour AdS/CFT

In this section we wish to compute the bulk to bulk correlators in the Kruskal formalism of AdS/CFT. For simplicity consider correlation functions of disturbances along an infinitely long string with fixed endpoints stretching from the horizon to the boundary. The action for these fluctuations has been given in eq. (3.3). We will consider the generating function

$$Z[J_1, J_2] = \int [\mathbb{D}x_1][\mathbb{D}x_2]\, e^{iS_1 - iS_2}\, e^{i\int \mathrm{d}t_1 \mathrm{d}r_1 J_1(t_1,r_1)\, x_1(t_1,r_1)}\, e^{-i\int \mathrm{d}t_2 \mathrm{d}r_2\, J_2(t_2,r_2)\, x_2(t_2,x_2)}\,, \quad (5.1)$$

where as before $[\mathbb{D}x_1]$ indicates a bulk path integral. We wish to compute all bulk two point functions

$$[iG(\omega, r, \bar{r})]_{ss'} \equiv \int \mathrm{d}t\, e^{i\omega t}\, [iG(t, r, \bar{r})]_{ss'}\ , \quad (5.2)$$

with

$$[iG(t - \bar{t}, r, \bar{r})]_{ss'} \equiv \frac{1}{i^2} \frac{\delta^2 \ln Z}{\delta J_s(t, r)\delta J_{s'}(\bar{t}, \bar{r})} \equiv i \begin{bmatrix} G_{11} & -G_{12} \\ -G_{21} & G_{22} \end{bmatrix} . \quad (5.3)$$

To this end we note that the classical equations of motion are

$$\left[\partial_r\big(T_o(r)\partial_r x(\omega, r)\big) + \frac{m\omega^2}{f} x(\omega, r) + J(\omega)\right] = 0\,, \quad (5.4)$$

where $J(\omega)$ is the external force per unit length on the string. Our task is to construct a Green function of this differential equation

$$\left[\partial_r\big(T_o(r)\partial_r G(r, \bar{r})\big) + \frac{m\omega^2}{f} G(r, \bar{r})\right] = \delta(r - \bar{r})\,. \quad (5.5)$$

Given two solutions of the differential equation $g_>(r)$ and $g_<(r)$ the Green function is easily constructed

$$G(r, \bar{r}) = \frac{g_>(r)g_<(\bar{r})\theta(r, \bar{r}) + g_<(r)g_>(\bar{r})\theta(\bar{r}, r)}{T_o(\bar{r})W(\bar{r})}\,, \quad (5.6)$$

where the Wronskian is

$$W(\bar{r}) = g'_>(\bar{r})g_<(\bar{r}) - g'_<(\bar{r})g_>(\bar{r})\,. \quad (5.7)$$

In the present case we wish to construct the Green function in the full Kruskal plane. We should find two solutions which obey the boundary conditions. These boundary conditions are that $g_>(r)$ should be normalizable as $r \to \infty$ in the right quadrant of the Kruskal plane. Similarly $g_<(r)$ should be normalizable as $r \to \infty$ in the left quadrant of the Kruskal plane. In formulas

$$g_>(r_1) = \mathrm{Im} F_\omega(r_1) \qquad \text{(right quadrant)} \ , \tag{5.8}$$

and

$$g_<(r_2) = \mathrm{Im} F_\omega(r_2) \qquad \text{(left quadrant)} \ . \tag{5.9}$$

Using the boundary conditions of the real time AdS/CFT we extend across the horizon in the Kruskal plane. Writing $\mathrm{Im}F(r_1) = (F(r_1) - F^*(r_1))/2i$ and analytically continuing $g_>$ according to eq. (4.8) we have

$$g_>(r_2) = \frac{1}{2i}\left[e^{-\sigma\omega} F_\omega(r_2) - e^{\omega/T} e^{-\sigma\omega} F^*_\omega(r_2)\right] \qquad \text{(left quadrant)}\,. \tag{5.10}$$

Similarly we realize that the analytic continuation of $g_<(r_2)$ to r_1 is

$$g_<(r_1) = \frac{1}{2i}\left[e^{+\sigma\omega} F_\omega(r_1) - e^{-\omega/T} e^{\sigma\omega} F^*_\omega(r_1)\right] \qquad \text{(right quadrant)}\,. \tag{5.11}$$

Now different correlators can be evaluated in a straightforward manner. Evaluating $G_{11}(r_1, \bar{r}_1)$ using eq. (5.6), eq. (5.8), and eq. (5.11), and for simplicity taking $r_1 > \bar{r}_1$, we have

$$\begin{aligned} G_{11}(r_1, \bar{r}_1) &= \frac{g_>(r_1) g_<(\bar{r}_1)}{W(\bar{r}_1)}\,, \\ &= \mathrm{Re} G_{\mathrm{ret}}(r_1, \bar{r}_1) + i(1+2n) \mathrm{Im} G_{\mathrm{ret}}(r_1, \bar{r}_1)\,. \end{aligned} \tag{5.12}$$

Here $n(\omega) = 1/(e^{\omega/T} - 1)$ is the thermal occupancy factor and $G_{\mathrm{ret}}(r, \bar{r})$ is given in eq. (3.13). In deriving this result we note the intermediate results

$$W(\bar{r}_1) = e^{+\omega\sigma}(1 - e^{-\omega/T}) \frac{1}{2i} W_{\mathrm{ret}}(\bar{r}_1)\,, \tag{5.13a}$$

$$W(\bar{r}_2) = e^{-\omega\sigma}(e^{+\omega/T} - 1) \frac{1}{2i} W_{\mathrm{ret}}(\bar{r}_2)\,, \tag{5.13b}$$

and the much used identity $n(-\omega) = -(1 + n(\omega))$.

Continuing in this way, we summarize these results for the bulk to bulk correlators

$$iG_{11}(r, \bar{r}) = +\, i \mathrm{Re}\, G_{\mathrm{ret}}(r, \bar{r}) - (1+2n) \mathrm{Im}\, G_{\mathrm{ret}}(r, \bar{r})\,, \tag{5.14a}$$

$$iG_{22}(r, \bar{r}) = -\, i \mathrm{Re}\, G_{\mathrm{ret}}(r, \bar{r}) - (1+2n) \mathrm{Im}\, G_{\mathrm{ret}}(r, \bar{r})\,, \tag{5.14b}$$

$$iG_{12}(r, \bar{r}) = -\, 2n\, e^{+\omega\sigma} \mathrm{Im}\, G_{\mathrm{ret}}(r, \bar{r})\,, \tag{5.14c}$$

$$iG_{21}(r, \bar{r}) = -\, 2(1+n)\, e^{-\omega\sigma} \mathrm{Im}\, G_{\mathrm{ret}}(r, \bar{r})\,. \tag{5.14d}$$

These are the familiar spectral and KMS relations between the retarded correlator and various contour correlation functions. It is reassuring that these thermal relations between the bulk to bulk correlators arise so easily in the Kruskal formalism although the result is not particularly new or surprising [29, 30].

To show that these correlation functions are indeed correctly normalized, one can shift the fields in the usual way

$$x_s(\omega, r) = \delta x_s(\omega, r) - \int \mathrm{d}\bar{r}\ \left[iG(\omega, r, \bar{r})\right]_{ss'} J_{s'}(\omega, \bar{r})\,, \tag{5.15}$$

and determine generating functional

$$Z[J_1, J_2] = Z[0,0] \exp\left(-\frac{1}{2}\int \frac{\mathrm{d}\omega}{2\pi}\mathrm{d}r\mathrm{d}\bar{r}\; J_s(-\omega, r)\, [iG(\omega, r, \bar{r})]_{ss'}\, J_{s'}(\omega, \bar{r})\right). \tag{5.16}$$

5.2 Bulk picture of Brownian motion

We will now develop a bulk picture of the Brownian motion. Rather than integrating out the entire bulk to determine an effective action for the boundary point, we will introduce a stretched horizon and integrate out only the fields which are inside the stretched horizon,

$$r_h \equiv 1 + \epsilon\,. \tag{5.17}$$

The path integral for the fluctuations of the string is

$$Z = \int \left[\mathrm{D}x_1^o\, \mathbb{D}x_1\, \mathrm{D}x_1^h\right] \left[\mathrm{D}x_2^o\, \mathbb{D}x_2\, \mathrm{D}x_2^h\right] \left[\mathbb{D}x_1^\epsilon\, \mathbb{D}x_2^\epsilon\right] e^{iS_1 - iS_2} e^{iS_1^\epsilon - iS_2^\epsilon}\,. \tag{5.18}$$

As before for the right quadrant for example, $\mathrm{D}x_1^o$ is the temporal path integral of the boundary endpoint of the string, $\mathbb{D}x_1$ indicates the bulk path integral of the string, $\mathrm{D}x^h$ denotes the temporal path integral of the string point on the stretched horizon, and $\mathbb{D}x_1^\epsilon$ is the bulk path integral inside of the stretched horizon. S_1 is the action integrated outside of the stretched horizon while S_1^ϵ is the action integrated inside the stretched horizon.

In a Gaussian approximation the integrals over the bulk coordinates inside the stretched horizon can be done. The result is a field independent determinant times the exponential of the action evaluated with the classical solution which passes through $x_1^h(\omega)$ and $x_2^h(\omega)$. This solution should also obey the contour boundary conditions given above. Substituting a classical solution into the action reduces to a boundary term at the horizon which gives an effective action for the horizon dynamics

$$S^h_{\text{eff}} = -\frac{T_o(r_h)}{2}\int_{r_1^h} x_1^\epsilon(-\omega, r)\, \partial_r x_1^\epsilon(\omega, r)\, \frac{\mathrm{d}\omega}{2\pi} + \frac{T_o(r_h)}{2}\int_{r_2^h} x_2^\epsilon(-\omega, r)\, \partial_r x_2^\epsilon(\omega, r)\, \frac{\mathrm{d}\omega}{2\pi}\,. \tag{5.19}$$

Going through the same procedure outlined above we have the same structure for the effective horizon action

$$iS^h_{\text{eff}} = -i\int \frac{\mathrm{d}\omega}{2\pi}\, x_a^h(-\omega)\left[G_R^h(\omega)\right] x_r^h(\omega) - \frac{1}{2}\int \frac{\mathrm{d}\omega}{2\pi} x_a^h(-\omega)\left[-(1+2n)\mathrm{Im}G_R^h(\omega)\right] x_a^h(\omega)\,, \tag{5.20}$$

but the retarded correlator is to be evaluated and normalized at the horizon. Using the near horizon behavior of the retarded solution $F_\omega(r) \sim (1 - 1/r^4)^{-i\omega/(4\pi T)}$ and $T_o(r)$ from eq. (3.4) we find

$$\begin{aligned} G_R^h(\omega) &= \frac{T_o(r)}{|F(\omega, r)|^2} F_{-\omega}(r)\partial_r F_\omega(r)\bigg|_{r=r_h}\,, \\ &= -\, i\omega\eta\,. \end{aligned} \tag{5.21}$$

The full partition function is now

$$Z = \int \left[\mathrm{D}x_1^o\, \mathbb{D}x_1\, \mathrm{D}x_1^h\right] \left[\mathrm{D}x_2^o\, \mathbb{D}x_2\, \mathrm{D}x_2^h\right] e^{iS_1 - iS_2} e^{iS^h_{\text{eff}}}\,. \tag{5.22}$$

Now in eq. (5.22) we have the two halves of the Kruskal plane coupled by the effective action of the horizon. We will show that the effect of this coupling is to give rise to thermal noise in the bulk. We rewrite the action in the ra basis[2]

$$x_r(\omega,r) \equiv \frac{x_1(r,\omega)+x_2(r,\omega)}{2}\,, \qquad x_a(\omega,r) \equiv x_1(r,\omega)-x_2(r,\omega)\,, \tag{5.23}$$

and the action in the bulk is

$$iS_1 - iS_2 = -i\int \frac{\mathrm{d}\omega}{2\pi}\mathrm{d}r\,\left[T_o(r)\partial_r x_a(-\omega,r)\,\partial_r x_r(\omega,r) - \frac{m\omega^2 x_r(\omega,r)x_a(-\omega,r)}{f}\right]\,. \tag{5.24}$$

We also follow what is now a standard procedure by introducing a horizon noise

$$e^{-\frac{1}{2}\int\frac{\mathrm{d}\omega}{2\pi}x_a^h(-\omega)\,[(1+2n)\omega\eta]\,x_a^h(\omega)} = \int D\xi^h e^{+i\int\xi^h(\omega)x_a^h(-\omega)}e^{-\frac{1}{2}\int\frac{\mathrm{d}\omega}{2\pi}\frac{\xi^h(-\omega)\xi^h(\omega)}{[(1+2n)\omega\eta]}}\,, \tag{5.25}$$

with the associated statistics

$$\left\langle \xi^h(-\omega)\xi^h(\omega)\right\rangle = (1+2n)\omega\eta\,. \tag{5.26}$$

We now integrate by parts and obtain two boundary terms, one terminating at the radius of the string endpoint r_m, and one terminating at the stretched horizon $r_h = 1+\epsilon$

$$\begin{aligned} iS_1 - iS_2 + iS^h_{\text{eff}} = &- i\int_{r_m}\frac{\mathrm{d}\omega}{2\pi}\,x_a^o(-\omega)\,[T_o(r_m)\partial_r x_r(\omega,r)] \\ &- i\int_{r_h}\frac{\mathrm{d}\omega}{2\pi}\,x_a^h(-\omega)\,\left[-T_o(r_h)\partial_r x_r(\omega,r) - i\omega\eta x_r^h(\omega) - \xi^h(\omega)\right] \\ &- i\int\frac{\mathrm{d}\omega}{2\pi}\mathrm{d}r\,x_a(-\omega,r)\,\left[-\partial_r\left(T_o(r)\partial_r x_r(\omega,r)\right) - \frac{m\omega^2\,x(\omega,r)}{f}\right]\,. \end{aligned} \tag{5.27}$$

Finally integrating over $x_a^o(-\omega)$, $x_a(-\omega,r)$, and $x_a^h(-\omega)$ in eq. (5.22) we are left with a set of stochastic equations as in the simple example given in section 2.2. We will record these in the next section.

5.3 Analysis and discussion

The preceding analysis leads to three equations

1. The boundary endpoint of the string obeys the deterministic equation

$$-\,T_o(r_m)\partial_r x_r(\omega,r) = 0\,. \tag{5.28}$$

   This is simply the Neumann boundary condition for the free end of the string.

2. The body of the string obeys the bulk equations of motion

$$\left[\partial_r\left(T_o(r)\partial_r x_r(\omega,r)\right) + \frac{m\omega^2}{f}x_r(\omega,r)\right] = 0\,. \tag{5.29}$$

[2]The label r for retarded should not be confused with the radial coordinate.

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3. Finally, the horizon endpoint obeys the stochastic equation of motion

$$T_o(r_h)\partial_r x_r(\omega, r) + \xi^h(\omega) = -i\omega\eta x_r^h(\omega)\,, \qquad \left\langle \xi^h(-\omega)\xi^h(\omega)\right\rangle = \eta\omega(1+2n)\,, \quad (5.30)$$

with $n(\omega) = 1/(\exp(\omega/T) - 1)$. Here we remind that $T_o(r) = (\sqrt{\lambda}\pi^2T^3/2)fr^4$ is the local tension in the string. The meaning of this equation is that the motion of the horizon endpoint $x_r^h(t)$ is overdamped. The resistance $-\eta\dot{x}^h$ exactly balances the applied forces which in this case are the pulling due to the string outside the horizon $T_o(r_h)\partial_r x_r$ and the random force $\xi^h(\omega)$ which comes from integrating out modes behind the horizon.

We now have determined a stochastic equation of motion for the endpoint of the string on the stretched horizon. These stochastic fluctuations are transmitted by the dynamics of the string to the boundary. We should show that the boundary endpoint of the string obeys the expected equation of motion in its most general form, eq. (2.22). We should further show that the symmetrized two point functions obey the expected dynamics previously computed using the Kruskal extension of AdS/CFT

$$\begin{aligned} \langle \Delta x_r(-\omega, r)\Delta x_r(\omega, \bar{r})\rangle &= -(1+2n)\mathrm{Im}G_{\mathrm{ret}}(\omega, r, \bar{r})\,, \\ &= (1+2n)\frac{\mathrm{Im}F_\omega(r)\mathrm{Im}_\omega F(\bar{r})}{-\mathrm{Im}G_R(\omega)}\,. \end{aligned} \quad (5.31)$$

First consider the average motion of the string. Averaging over the noise we find the average string coordinates obey the usual equations of motion together with retarded boundary conditions

$$\eta\left[4\pi T(1-1/r)\partial_r \langle x(\omega, r)\rangle + i\omega\left\langle x^h\right\rangle\right] = 0\,, \quad (5.32)$$

where we have used the definition for $T_o(r)$. This equation together with the fact that $\langle x_r\rangle$ is a solution says that the average obeys the retarded boundary conditions, i.e. behaves as $(1-1/r)^{-i\omega/4\pi T}$.

Next consider the behavior near the boundary of AdS $r \to r_m$. Here the equations of motion guarantee that the solution is a superposition of the non-normalizable mode and the normalizable mode which we can choose to be $F_\omega(r)$ and $\mathrm{Im}F_\omega(r)$ respectively

$$x(\omega, r) = x_o(\omega)F_\omega(r) + \xi^o(\omega)\left[\frac{\mathrm{Im}F_\omega(r)}{-\mathrm{Im}G_R(\omega)}\right]\,. \quad (5.33)$$

We also have specified the boundary value of the non-normalizable mode $x_o(\omega)$ and have recognized that $\xi^o(\omega)$ must be a stochastic variable since the retarded solution reproduces the average motion. We have chosen to divide $\mathrm{Im}F_\omega(r)$ by $-\mathrm{Im}G_R(\omega)$ so that the $\xi^o(\omega)$ has the interpretation as the random force on the AdS boundary as we will show now.

Plugging this functional form eq. (5.33) into the Neumann boundary conditions

$$T_o(r)\partial_r x_r|_{r=r_m} = 0\,, \quad (5.34)$$

yields the expected equation of motion for the endpoint

$$\left[-M_Q\omega^2 + G_R(\omega)\right] x_o(\omega) = \xi^o(\omega)\,. \quad (5.35)$$

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Our task now is to show that $\langle \xi^o(-\omega)\xi^o(\omega)\rangle$ obeys the fluctuation-dissipation relation

$$\langle \xi^o(-\omega)\xi^o(\omega)\rangle = -(1+2n)\mathrm{Im}G_R(\omega)\,. \tag{5.36}$$

Returning to the horizon we have the stochastic equation of motion

$$T_o(r_h)\partial_r x_r(\omega, r) + \xi^h(\omega) = -i\omega\eta x_r^h(\omega)\,. \tag{5.37}$$

Substituting eq. (5.33) into this equation we solve for $\xi^o(\omega)$ in terms of $\xi^h(\omega)$. Our task is simplified by first recognizing the origin of the first term on the right hand side

$$-i\omega\eta = T_o(r_h)\frac{F_{-\omega}(r_h)\partial_r F_\omega(r_h)}{|F_\omega(r_h)|^2}\,, \tag{5.38}$$

so that the equation for ξ^o resulting from eq. (5.37) is

$$\frac{\xi^o(\omega)}{-\mathrm{Im}G_R(\omega)}T_o(r_h)\left[F_\omega(r_h)\partial_r\mathrm{Im}F_\omega - \mathrm{Im}F(r_h)\partial_r F_\omega(r_h)\right] + F_\omega(r_h)\xi^h(\omega) = 0\,. \tag{5.39}$$

The term in square brackets on the left hand side is the Wronskian of the two solutions and when multiplied by $T_o(r)$ equals $+\mathrm{Im}G_R(\omega)$ and we find

$$\xi^o(\omega) = F_\omega(r_h)\xi^h(\omega)\,. \tag{5.40}$$

Averaging according to the horizon statistics in eq. (5.30), recognizing its origin eq. (5.38), and again using the Wronskian relation eq. (3.12), we determine the statistics of ξ^o

$$\langle \xi^o(-\omega)\xi^o(\omega)\rangle = -(1+2n)\mathrm{Im}G_R(\omega)\,. \tag{5.41}$$

Thus random force in the boundary theory obeys the expected statistics of the fluctuation-dissipation theorem.

Finally we would like to compute the bulk two point functions

$$\langle \Delta x_r(-\omega, r)\Delta x_r(\omega, \bar r)\rangle\,, \tag{5.42}$$

where $\Delta x_r(\omega, r)$ denotes the deviation from the behavior in the bulk due to the motion on the boundary $\Delta x_r \equiv x_r(\omega, r) - x_r^o(\omega)F_\omega(r)$. This is straightforward using the decomposition in eq. (5.33), and the boundary statistics in eq. (5.41); the result is

$$\begin{aligned}\langle \Delta x_r(-\omega, r)\Delta x_r(\omega, \bar r)\rangle &= (1+2n)\frac{\mathrm{Im}F_\omega(r)\mathrm{Im}F_\omega(\bar r)}{-\mathrm{Im}G_R(\omega)}\,, &(5.43)\\ &= -(1+2n)\mathrm{Im}G_{\mathrm{ret}}(\omega, r, \bar r)\,. &(5.44)\end{aligned}$$

This result is naturally the same as computed previously using the contour correlation function in AdS/CFT. It is also neatly consistent with the bulk fluctuation-dissipation theorem.

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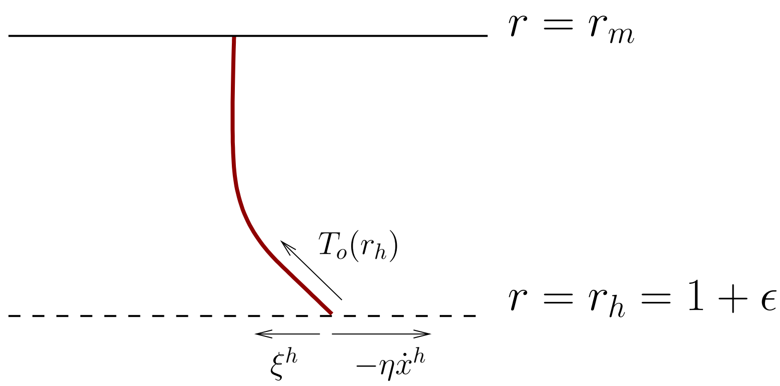

Figure 4. Balance of forces on the stretched horizon. The resistive force $-\eta\dot{x}^h$ precisely balances the random force ξ^h and the tension T_o leading to overdamped motion.

6 Summary and the physical picture

6.1 Summary

In the previous section we have shown that a stochastic boundary condition emerges on the stretched horizon after integrating out the fluctuations inside this surface (see eq. (5.30))[3]

$$T_o(r_h)\partial_r x(t, r_h) + \xi^h(t) = \eta\, \dot{x}^h(t) \qquad \left\langle \xi^h(t)\xi^h(t') \right\rangle = G^h_{\rm sym}(t - t')\,. \tag{6.1}$$

Here η is the drag of the horizon (or the late time drag of the quark), ξ^h is the random force on the horizon endpoint, and $\xi^h(\omega)$ obeys the *horizon* fluctuation dissipation theorem

$$G^h_{\rm sym}(\omega) = +(1 + 2n)\omega\eta\,. \tag{6.2}$$

This equation has a simple physical interpretation illustrated in figure 4. The motion of the horizon $x^h(t)$ is overdamped, *i.e.* the resistive force $-\eta\dot{x}^h$ exactly balances the string force $T_o(r_h)\partial_r x(t, r_h)$ and the random horizon force ξ^h. There is no transverse acceleration.

This stochastic force on the horizon when transmitted to the boundary leads to an equation of motion for the endpoint of the string

$$M_Q \frac{d^2x^o}{dt^2} + \int^t G_R(t - t')x^o(t') = \xi^o(t)\,, \qquad \left\langle \xi^o(t)\xi^o(t') \right\rangle = G_{\rm sym}(t, t')\,. \tag{6.3}$$

which is the expected generalized Langevin equation in eq. (2.23). Here $G_{\rm sym}(\omega)$ obeys *boundary* the fluctuation dissipation relation

$$G_{\rm sym}(\omega) = -(1 + 2n){\rm Im}G_R(\omega)\,, \tag{6.4}$$

and $G_R(\omega)$ is the usual retarded force-force propagator computed using AdS/CFT. The random force on the boundary $\xi^o(t)$ is directly related to the random force on the horizon

[3]In this section the retarded r label (as in x_r) is understood.

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ξ^h. Denoting $F(t,r)$ the usual retarded boundary to bulk propagator, *i.e.* $F_\omega(r_m) = 1$ on the boundary and $F_\omega \sim (1-1/r)^{-i\omega/4\pi T}$ near the horizon, the relation between the horizon and boundary stochastic forces is (see eq. (5.40))

$$\xi^o(t) = \int^t \mathrm{d}t'\, F(t-t', r_h)\, \xi^h(t')\,, \tag{6.5}$$

It takes a time of order $\sim 1/(\pi T)\log(1/\epsilon)$ for the noise from the horizon to reach the tip of the string. ϵ should be considered small but not exponentially small so this time scale is really $1/\pi T$.

We also have a picture of the fluctuations in the bulk. The coordinate of the string in the bulk are given by two pieces (see eq. (5.33))

$$x(t,r) = \int^t \mathrm{d}t' F(t-t',r)x^o(t') + \Delta x(t,r)\,, \tag{6.6}$$

which reflect the retarded response to the boundary motion $x^o(t')$ and a deviation. The deviation $\Delta x(t,r)$ is a random variable obeying the statistics (see eq. (5.43))

$$\left\langle \Delta x(t,r)\Delta x(t',r')\right\rangle = G_{\rm sym}(t-t',r,r')\,, \tag{6.7}$$

where $G_{\rm sym}(t-t',r,r')$ is the symmetrized bulk to bulk correlator. This correlator was computed using the Kruskal extension of the AdS/CFT and is related to the imaginary part of the retarded bulk to bulk correlator according to a *bulk* fluctuation dissipation theorem

$$G_{\rm sym}(\omega,r,r') = -(1+2n)\,{\rm Im}G_{\rm ret}(\omega,r,r') = (1+2n)\,\frac{{\rm Im}F_\omega(r){\rm Im}F_\omega(r')}{-{\rm Im}G_R(\omega)}\,. \tag{6.8}$$

The explicit form for the fluctuation amplitude in frequency space is

$$\Delta x(\omega,r) = \xi^o(\omega)\left[\frac{{\rm Im}F_\omega(r)}{-{\rm Im}G_R(\omega)}\right]\,, \tag{6.9}$$

where $\xi^o(\omega)$ is the boundary force.

6.2 The physical picture

Here we would like to consider the small frequency limit where an explicit analytic form for the retarded function F_ω is known and has a simple physical interpretation in terms of trailing strings. A quark in equilibrium moves quite slowly

$$v_{\rm th} \sim \sqrt{\frac{T}{M_{\rm kin}}} \sim \frac{1}{\lambda^{1/4}}\frac{1}{\sqrt{r_m}}\,, \tag{6.10}$$

but it takes a long time τ_R for this heavy quark to randomize its velocity

$$\tau_R \sim \frac{M_{\rm kin}}{\eta} \sim \frac{r_m}{T}\,. \tag{6.11}$$

(This can be seen by examining the Langevin equations and neglecting the noise.) The distance the quark moves over this relaxation time x_R is

$$x_R \sim v_{\rm th}\tau_R \sim \frac{1}{\lambda^{1/4}T}\sqrt{r_m}\,. \tag{6.12}$$

The dynamics that is observed depends on how the spatial $x_{\rm obs}$ and temporal resolution scales $\tau_{\rm obs}$ of the measurement compare to these scales x_R and τ_R.

First consider the time period over which quarks moves with nearly constant velocity v

$$\frac{1}{T} \ll \tau_{\rm obs} \ll \tau_R \,. \tag{6.13}$$

For a quark moving slowly on the boundary with constant velocity v it will trail behind it a trailing string at least on average

$$\langle x(t,r)\rangle = x_o(t) + v\Delta x_{\rm TS}(r)\,, \tag{6.14}$$

with

$$\Delta x_{\rm TS}(r) = \frac{1}{2\pi T}\left[\tan^{-1}(z) - \tanh^{-1}(z)\right]\,, \tag{6.15}$$

and $z \equiv 1/r$. Here we have used eq. (5.33) for retarded response to the boundary motion and the explicit form of $F_\omega(r)$ at small frequency, eq. (3.7). The term $\Delta x_{\rm TS}$ is the "trailing string" solution of ref. [17, 19]. The distance between the head of the quark $x^o(t)$ and the average body of string is of order

$$v\Delta x_{\rm TS} \sim \frac{v_{\rm th}}{T} \sim \frac{1}{\lambda^{1/4}T}\frac{1}{\sqrt{r_m}}\,. \tag{6.16}$$

This only gives the average behavior of the string. In general there is an additional random component which in the small frequency limit is white noise; using eq. (6.9) we have

$$\Delta x \equiv x(t,r) - \langle x(t,r)\rangle = \frac{-1}{\eta}\xi^o(t)\,\Delta x_{\rm TS}(r)\,, \qquad \left\langle \xi^o(t)\xi^o(t')\right\rangle = 2T\eta\delta(t-t')\,. \tag{6.17}$$

Thus we see that around this average trailing string there is a stochastic ensemble of trailing strings which flip-flop around the head of the quark. This is illustrated above in figure 5. To estimate the amplitude of this stochastic process, let us imagine implementing this process on the computer where one would take time step Δt and then the width of the Gaussian process would be $\langle \xi^o\xi^o\rangle = 2T\eta/\Delta t$. Taking the width Δt to be of order the memory time scale $1/T$, we estimate that the string fluctuates around the average trailing string by an amount

$$\sqrt{(\Delta x)^2} \sim \frac{1}{\sqrt{\eta}} \sim \frac{1}{\lambda^{1/4}T}\,, \tag{6.18}$$

This is larger than the average deviation $v\Delta x_{\rm TS}$ from the endpoint since it is not suppressed by $1/\sqrt{r_m}$. Thus the average trailing string is nearly straight and the noise consists of a flip flopping trailing string solution. Notice the minus appearing in front of the eq. (6.17). This is physically correct. When the random force on the boundary quark is positive, the string is out in front of the quark. When the random force is negative, the string trails behind the quark.

Now let us consider the ultimate long time limit where the time scales and spatial scales on which we are observing the quark are large compared to the relaxation time τ_R and relaxation length x_R

$$\tau_{\rm obs} \gg \tau_R\,, \qquad x_{\rm obs} \gg v_{\rm th}\tau_R \gtrsim \frac{1}{\lambda^{1/4}}\frac{1}{T}\sqrt{r_m}\,. \tag{6.19}$$

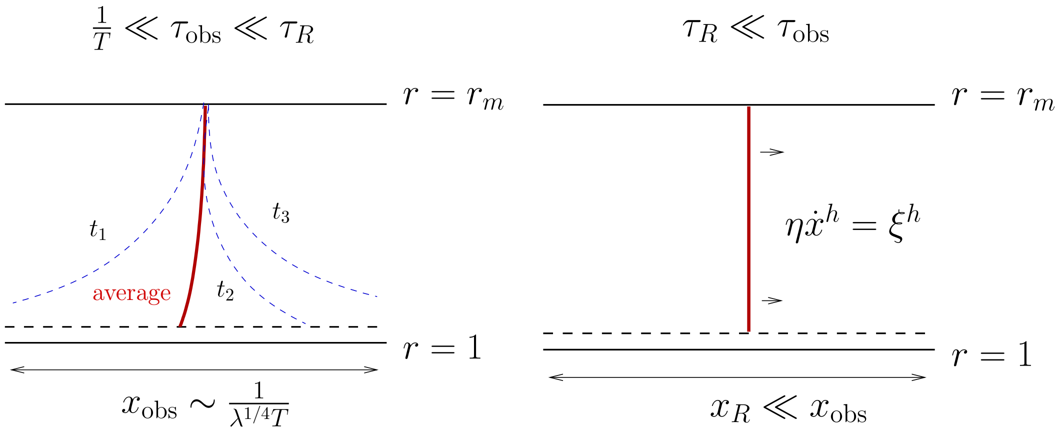

Figure 5. (a) The physical picture that emerges when observing the quark on relatively short time scales $1/T \ll \tau_{\rm obs} \ll \tau_R$. Here we show three subsequent time steps, t_1, t_2, t_3; at each time step the string fluctuates to a new "trailing string" giving rise to a random force on the boundary. The average trailing string is perceived as a drag. (b) The physical picture that emerges on very long time and spatial scales. The horizon diffuses and the string is brought along.

In this limit one can drop the influence of the bulk on the horizon dynamics, *i.e.* discard the $T_o \partial_r x$ in eq. (6.1) since it is averaged over many different boundary velocities over the observation time $\tau_{\rm obs} \gg \tau_R$. The string in bulk is perfectly straight on the spatial scales we are considering. Thus the equation of motion obeyed by the horizon is

$$\frac{\xi^h}{\eta} = \frac{dx^h}{dt} \, . \tag{6.20}$$

This is the overdamped diffusive limit discussed in section 2.1. The result is that the horizon endpoint and the boundary endpoint diffuse in lockstep according to the expected rate

$$\left\langle [x^h(t)]^2 \right\rangle = \frac{2T}{\eta} t \, . \tag{6.21}$$

In summary, we have exhibited the full structure of the thermal noise on a fluctuating string in AdS/CFT. It is hoped that this will lay the groundwork to study the fluctuations of gravitons and other fields using the correspondence. The challenge now is to use the real time formalism in a truly out of equilibrium setting such as studied in ref. [55].

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Note Added. In the final two weeks of preparation, a preprint appeared [56] which addressed some of the aspects studied here. In particular the stochastic boundary condition given in eq. (6.1) was determined and the basic conclusion that the horizon diffuses according to eq. (6.21) was also reached in this work. It is interesting that the derivation of the stochastic boundary condition in ref. [56] differs substantially from the presentation given here. Presumably by marrying the two derivations a deeper understanding of Hawking radiation will be reached.

A Small fluctuations and the trailing string

The purpose of this appendix is to establish notation and to collect prior results. The Nambu-Goto action is

$$S = -\frac{1}{2\pi\ell_s^2}\int d\tau d\sigma\,\sqrt{-\det h_{ab}}\,. \tag{A.1}$$

For small fluctuations we parameterize the string as

$$(\tau,\sigma)\mapsto(t=\tau, r=\sigma, x=x(t,r))\,, \tag{A.2}$$

determine the induced metric,

$$h_{tt} = \left[-fr^2 + r^2\dot{x}^2\right]L^2(\pi T)^2\,, \tag{A.3a}$$

$$h_{rr} = \left[\frac{1}{fr^2} + (\pi T)^2 r^2 (x')^2\right]L^2\,, \tag{A.3b}$$

$$h_{tr} = \left[(\dot{x})x' r^2(\pi T)^2\right]L^2\,, \tag{A.3c}$$

and write the action for small fluctuations as

$$S = -\frac{(\pi T)L^2}{2\pi\ell_s^2}\int dt dr\left[1 + \frac{1}{2}(\pi T)^2 fr^4(x')^2 - \frac{1}{2}\frac{\dot{x}^2}{f}\right]. \tag{A.4}$$

In the body of the text we write this as

$$S = -\int dt dr\left[m + \frac{1}{2}T_o(r)(\partial_r x)^2 - \frac{m}{2f}(\partial_t x)^2\right], \tag{A.5}$$

where the local tension is

$$T_o(r) = \frac{(\pi T)^3 L^2}{2\pi\ell_s^2}\,fr^4 = \left(\sqrt{\lambda}\pi^2T^3/2\right)\,fr^4\,, \tag{A.6}$$

and the mass per unit r

$$m = \frac{(\pi T)L^2}{2\pi\ell_s^2} = \frac{\sqrt{\lambda}T}{2}\,. \tag{A.7}$$

Then the equation of motion is

$$\frac{\mathfrak{w}^2}{f}x + \partial_r(fr^4\partial_r x) = 0\,. \tag{A.8}$$

where, as is customary, we have defined $\mathfrak{w} = \omega/(\pi T)$.

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This is a second order differential equation and there are two solutions. Near the horizon $r \to 1$ the solutions are either infalling (-) or outgoing (+)

$$\left(1 - \frac{1}{r^4}\right)^{\mp \frac{i\mathfrak{w}}{4}} . \tag{A.9}$$

The solutions near the boundary $r \to r_m$ consist of a normalizable and a non-normalizable mode. The retarded solution $F_\omega(r)$ is the solution which approaches one near the boundary and is infalling at the horizon. More specifically near the boundary F_ω behaves as

$$F_\omega(r) = \left(1 + \frac{\mathfrak{w}^2}{2r^2} + \dots\right) - \frac{B(\omega)}{3r^3}(1 + \dots) , \tag{A.10}$$

where the ellipses denote terms suppressed by additional powers of $1/r$. In evaluating the motion of the quark we often evaluate the combination

$$\lim_{r \to r_m} T_o(r) F_{-\omega} \partial_r F_\omega(r) = -M_Q^o \omega^2 + G_R(\omega) , \tag{A.11}$$

where $G_R(\omega) = (\sqrt{\lambda}\pi^2 T^3/2) B(\omega)$ is the retarded force-force correlator. The first term $M_Q^o \omega^2$ comes from the "divergent" $\mathfrak{w}^2/2r^2$ term of the real part of the retarded Green function and we have identified

$$M_Q^o = \frac{L^2(\pi T) r_m}{2\pi \ell_s^2} = \frac{\bar{r}_m}{2\pi \ell_s^2} , \tag{A.12}$$

as the zero-temperature mass of the quark where $\bar{r}_m$ refers to the "canonical" coordinates in eq. (3.1). That this is the zero temperature mass can also be seen from the leading term of eq. (A.4).

In general eq. (A.8) can not be solved exactly. However we can set up a perturbation expansion at small frequencies,

$$F_\omega(r) = (1 - 1/r^4)^{-i\mathfrak{w}/4} \left[1 - i\mathfrak{w} F_\omega^{(1)}(r) - \mathfrak{w}^2 F_\omega^{(2)}(r) + \dots\right] . \tag{A.13}$$

Substituting this ansatz into the equation of motion we end up with a hierarchy of differential equations. We solve these equations order by order in ω by demanding that the solution behaves as

$$F_\omega(r) = \left(1 - \frac{1}{r^4}\right)^{-i\mathfrak{w}/4} \times (\text{regular function at the horizon}) . \tag{A.14}$$

We find

$$F_\omega^{(1)}(r) = -\frac{1}{2}\ln(z+1) - \frac{1}{4}\ln\left(z^2+1\right) + \frac{1}{2}\arctan(z) , \tag{A.15}$$

where here and below $z \equiv 1/r$. At quadratic order we have

$$F_\omega^{(2)}(r) = \int_0^{\frac{1}{r}} dz \frac{z(1-z)\left[z\ln\left(z^2+1\right) - 2z\arctan(z) + 2\ln(z+1)z - 4\right]}{4(1-z^4)} . \tag{A.16}$$

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This solution has a simple physical interpretation. The leading order solution is multiplied by the position of the tip string $x_o(\omega)$. Provided we are not exponentially close to the horizon we can expand the leading $(1-z^4)^{-i\mathfrak{w}/4}$ factor. This yields the solution

$$x_o(\omega, r) \equiv x_o(\omega) F_\omega(r) = x_o(\omega) + v(\omega)\Delta x_{\mathrm{TS}} + a(\omega)\Delta x_a \,, \tag{A.17}$$

where $v(\omega) = -i\omega x_o(\omega)$ is the velocity of the endpoint, and $a(\omega) = (-i\omega)^2 x_o(\omega)$ is the acceleration of the endpoint. Further we have defined

$$\Delta x_{\mathrm{TS}}(z) = \frac{1}{2\pi T}\left[\tan^{-1}(z) - \tanh^{-1}(z)\right] \,, \tag{A.18a}$$

$$\Delta x_a(z) = \frac{1}{(\pi T)^2}\left[\frac{1}{32}\log^2(1-z^4) + \frac{1}{4}\log(1-z^4)F_\omega^{(1)} + F_\omega^{(2)}\right] \,. \tag{A.18b}$$

The leading term is given by Δx_{TS} which is simply the "trailing string" solutions of ref. [17, 19]. Naturally to leading order in the frequency of the sinusoidal oscillation one simply recovers that the string trails behind the head of the quark with the expected form.

The subleading term is described by the acceleration Δx_a. A graph of this function is given in figure 6 and has a simple interpretation. Consider sinusoidal oscillations: when the head of the quark is moving forward and undergoing negative acceleration, the body of the string travels ahead of the trailing string solution due to inertia. Thus $-\Delta x_a$ should be positive as $r \to \infty$ reflecting the fact that the displacement is 180^o out of phase with the acceleration. This inertial effect is indicated by the $1/2r^2$ curve in figure 6. The dynamics close to the horizon stems from expanding out the leading $(1-z^4)^{-i\mathfrak{w}/4}$ factor. The $(1-z^4)^{-i\mathfrak{w}/4}$ behavior near the horizon has the interpretation that the string endpoint on the stretched horizon is overdamped.

Using the solutions given above, we can expand these functions close to the boundary

$$F_\omega(r) = \left(1 + \frac{\mathfrak{w}^2}{2r^2} + \ldots\right) + \frac{(i\mathfrak{w} - \mathfrak{w}^2)}{3r^3}(1 + \ldots) \,, \tag{A.19}$$

and determine the retarded force-force correlator

$$T_o(r) F_{-\omega}\partial_r F_\omega = -M_Q^o\omega^2 + G_R(\omega) \,, \tag{A.20}$$

$$= -M_Q^o\omega^2 - i\omega\eta - \omega^2\Delta M \,. \tag{A.21}$$

Here we have defined the transport coefficient

$$\eta = \frac{1}{2}\sqrt{\lambda}\pi T^2 \,, \tag{A.22}$$

first computed in ref. [17–19] and the in-medium mass shift

$$\Delta M = -\frac{\sqrt{\lambda}T}{2} \,, \tag{A.23}$$

first computed in ref. [17]. The fact that the mass shift is negative stems from the overdamped motion of the horizon endpoint. Also used in the text is the kinetic mass

$$M_{\mathrm{kin}} \equiv M_Q^o + \Delta M \,. \tag{A.24}$$

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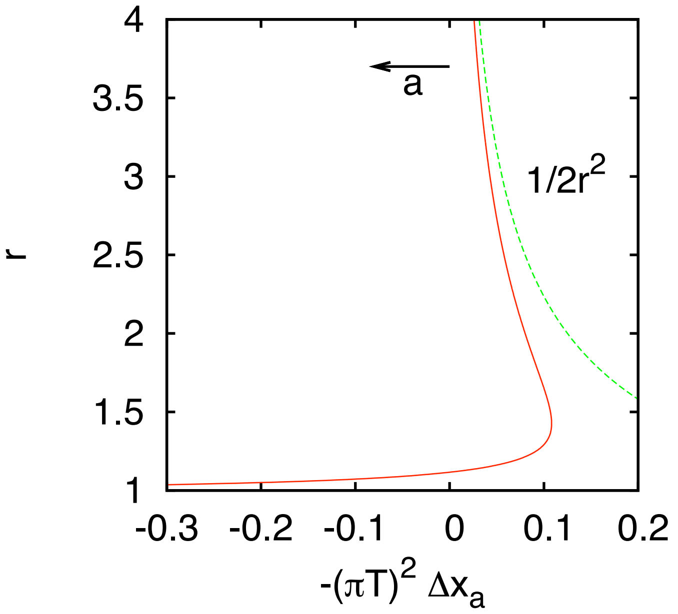

Figure 6. The deviation of the string position from the trailing string solution (in units of $|a(t)|/(\pi T)^2$) during slow sinusoidal acceleration of the boundary endpoint. Here the acceleration is negative and the quark is moving forward. More specifically we are plotting $-(\pi T)^2\Delta x_a$ given in the text.

B Notation for the Kruskal plane

In this appendix we establish notation for the Kruskal variables used in the body of the text. We first define $r_*(r)$

$$r_*(r) = \frac{1}{\pi T}\int^r \frac{dr}{f(r)r^2} = \frac{1}{2\pi T}\tan^{-1}(r) + \frac{1}{4\pi T}\ln(r-1) - \frac{1}{4\pi T}\ln(r+1)\,. \tag{B.1}$$

Then U and V are defined by the relations

$$t = \frac{1}{4\pi T}\log(V) - \frac{1}{4\pi T}\log(-U)\,, \tag{B.2a}$$

$$r_* = \frac{1}{4\pi T}\log(V) + \frac{1}{4\pi T}\log(-U)\,. \tag{B.2b}$$

The near horizon behaviors of $\nu_- \equiv t + r_*$ and $\nu_+ \equiv t - r_*$ are

$$\nu_- \simeq t + \frac{1}{4\pi T}\log(r-1)\,, \tag{B.3a}$$

$$\nu_+ \simeq t - \frac{1}{4\pi T}\log(r-1)\,. \tag{B.3b}$$

Also note that $-UV$ is a simple function of r

$$(-U)V = e^{4r_*\pi T} = \frac{r-1}{r+1}e^{2\tan^{-1}(r)}\,. \tag{B.4}$$

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